Cooperation in Repeated Prisoner's Dilemma with Outside Options*

by

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Abstract: In many repeated interactions, repetition is not guaranteed but instead must be agreed upon. We formulate a model of voluntary repetition by introducing outside options to a repeated Prisoner's Dilemma and investigate how the *structure* of outside options affects the sustainability of mutual cooperation. Under stochastic outside options, the optimal exit decision and the incentive to cooperate within the repeated game are interrelated. This is a contrast to most literature which deals with either exit decision only (search models etc.) or cooperation incentive only (repeated games, contracts etc.). We show that the mean of outside options has a monotone negative effect on cooperation, while perturbation of the option values may enhance cooperation, when the player wants to wait for a very good option to realize. Two-sided options weaken the perturbation effect because one may end up with a low option and thus becomes less patient. (145 words)

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1 Introduction

In many repeated interactions, repetition is not guaranteed but instead must be agreed upon. Workers can quit, customers can walk away, and couples can break up. If it is possible to strategically exit from a repeated interaction, the ordinary repeated-game framework no longer applies. Ordinary repeated games assume that the same set of players play the same stage game repeatedly for a fixed (possibly infinite) length of time. Therefore no player has a choice to exit from the game. At the other extreme, random matching games[1] assume that in every period a player is randomly matched with a new partner. Therefore no player has a choice to continue the game with the same partner. However, many economic situations are in an intermediate case where players can play a game repeatedly, but they can also terminate the interaction. There is a growing literature on these "endogenously repeated" games.

In this literature, three issues have been mainly analyzed. First, ordinary trigger strategies do not constitute an equilibrium since cooperation from the beginning of a relationship is vulnerable to defection and running away. Instead, gradual cooperation or trust-building strategy becomes an equilibrium. (Datta, 1996; Kranton, 1996a; Fujiwara-Greve, 2002 and Fujiwara-Greve and Okuno-Fujiwara, 2009.) Second, gradual cooperation is also useful in incomplete information models to sort out the types of players. (Ghosh and Ray, 1996; Kranton, 1996a,b; Watson, 2002 and Furusawa and Kawakami, 2008.) Third, a modified folk theorem holds with appropriate lower bounds of the equilibrium payoffs. (Casas-Arce, 2007 and Yasuda, 2007.)

We add a new angle to the analysis of the endogenously repeated games by looking at the interaction between in-game behavior and what a player may receive outside of the game. In the literature, often the outside structure of a game is fixed and the analysis is focused on in-game strategic outcomes given the outside structure. Even in the above-mentioned endogenously repeated game literature, the continuation payoff after termination of a partnership is unique. By contrast in other research fields such as search theory and operations research, the main interest lies in the effect of outside structural changes on individual behavior/decision-making, but the strategic interaction among decision-makers is omitted. In this paper we consider strategic interaction of two players under varying outside *structures* of the game.[2]

[1] See for example, Kandori (1992), Ellison, (1994), and Okuno-Fujiwara and Postlewaite (1995).

[2] Casas-Arce (2007) also considers variations of outside option *values* as well as not only quitting but also firing the opponent. His focus is the use of firing and quitting options as punishment to

P1 \ P2	C	D
C	5, 5	−10, 7
D	7, −10	0, 0

Table 1: An Example

Specifically, we examine variants of repeated Prisoner's Dilemma from which players can exit by taking an outside option and investigate effects of outside option *structure* on the sustainability of cooperation. It turns out that the "locked-in" feature of ordinary repeated game is a very strong cooperation enforcement system. The existence of a relevant outside option (greater than the in-game punishment payoff) increases the necessary level of discount factor to sustain cooperation as compared to the one in ordinary repeated games, and in some cases for any discount factor cooperation is not possible. However, within the outside option model, the relative difficulty of repeated cooperation is dependent on the structure of outside options. In particular, if the option values are uncertain, in some cases it is easier to sustain repeated cooperation than when they are certain. Therefore, perturbation of outside options is not always bad for cooperation.

Let us give an example to explain the logic. In each period, as long as the two players are in the game, they play the Prisoner's Dilemma of Table 1. After playing the Prisoner's Dilemma, an outside option is available to Player 1. Player 2 has no such option. The game repeats (Prisoner's Dilemma and then the outside option to exit) as long as Player 1 does not take the outside option, and each player maximizes the total expected discounted payoff with a discount factor $\delta \in (0,1)$. Suppose that, in any period, the outside option is the same and unique, and it gives the total payoff of $4/(1-\delta)$ to Player 1 after exit. Player 2's payoff after Player 1 ends the game is normalized to be zero.

Note that if the game is an ordinary repeated game without the outside option, the infinitely repeated cooperation $(C,C),(C,C),\ldots$ (which we call *the eternal cooperation*) is sustainable by the grim trigger strategy if

$$\frac{5}{1-\delta} \geqq 7 + \delta\frac{0}{1-\delta} \iff \delta \geqq \frac{2}{7} \approx 0.285.$$

However, if the outside option of the value $4/(1-\delta)$ is available, Player 1's contin-

sustain cooperation in fixed length repeated games. Thus our focus is complementary to his.

uation value after choosing D is increased to $4/(1-\delta)$. Therefore, Player 1 may not follow the eternal cooperation $(C,C),(C,C),\ldots$ even if δ is not so small. For example, when $\delta = 0.6$,

$$\frac{5}{1-\delta} = 12.5 < 13 = 7 + \delta\frac{4}{1-\delta}.$$

This illustrates that the existence of an outside option greater than the in-game punishment payoff creates difficulty in achieving cooperation, in the sense that the range of discount factors that sustain repeated cooperation shrinks.

Next, suppose that, at the end of each period, Player 1 has two possible outside options of the form $(4+\alpha)/(1-\delta)$ and $(4-\alpha)/(1-\delta)$ (where $\alpha > 0$), and these arrive with equal probability. The average outside option is $4/(1-\delta)$. When α is small (i.e., less than 1), then there is no point of taking any of the outside options if players are to repeat (C,C). When α is large enough, however, the better outside option exceeds the payoff from the repeated (C,C) so that the infinitely repeated cooperation becomes impossible for any δ. However, Player 1 may cooperate until she receives the better option. We call this play path *stochastic cooperation*. Let us compute the total expected discounted payoff of cooperation until the better option arrives. Let V be the continuation value at the end of a period, before an option realizes. Then the total expected payoff of repeating (C,C) until $(4+\alpha)/(1-\delta)$ arrives is of the form $5+\delta V$, where the continuation value V satisfies the following recursive equation.

$$V = \frac{1}{2}\cdot\frac{4+\alpha}{1-\delta} + \frac{1}{2}(5+\delta V).$$

For example, when $\alpha = 1.5$ and $\delta = 0.6$, then $V \approx 13.39$, and the value of the stochastic cooperation is $5+\delta V \approx 13.0357$. This is greater than the value of the eternal cooperation, $12.5 = 5/(1-\delta)$.

The value of one-shot deviations also needs to be checked more carefully. The optimal exit strategy for Player 1 is either to exit immediately by taking any option or to wait for $(4+\alpha)/(1-\delta)$. If she deviates and then waits for the good option while suffering from the punishment payoff of 0 in the stage game, the total expected payoff is of the form $7+\delta W$, where the continuation value W satisfies

$$W = \frac{1}{2}\cdot\frac{4+\alpha}{1-\delta} + \frac{1}{2}(0+\delta W).$$

Thus $7+\delta W \approx 12.89$ for $\alpha = 1.5$ and $\delta = 0.6$. If Player 1 defects and then exits immediately by taking any option, the expected payoff is $7+\delta\frac{4}{1-\delta} = 13$ as before.

Therefore, in this example, it is optimal to exit immediately after a deviation. However, $5 + \delta V > 13 = 7 + \delta \frac{4}{1-\delta}$ implies that $\delta = 0.6$ sustains the stochastic cooperation, although it does not sustain any cooperation when the outside option is $4/(1-\delta)$ for sure.

The above example shows that the structure of outside options makes a difference in sustaining cooperation for mid-range discount factors. In addition, given a discount factor and the mean of the outside options, we can investigate how the spread α affects the sustainability of cooperation. In this example, when α is small, no cooperation is possible, just like in the unique option case. As α increases, the value of cooperation while waiting for the good option increases so that stochastic cooperation becomes an equilibrium behavior.[3] This can be generalized for a mid-range of δ and for a class of general distributions of outside options. Therefore perturbations of outside options may enhance cooperation.

The repeated game literature, however, so far found negative effects of perturbations on cooperation. Rotemberg and Saloner (1986) perturb payoffs of the stage game, while Baye and Jansen (1996) and Dal Bó (2007) perturb the discount factor.[4] In these models the optimal (eternal) cooperation *levels* are shown to be lower than the one in the absence of perturbation. Although they did not investigate the lower bound of the discount factors by fixing a level of cooperation, it would be greater than the one under no perturbation. This is clarified in Yasuda and Fujiwara-Greve (2009).

The key to these negative results is that, in ordinary repeated games, when the perturbation creates difficulty to cooperate (a high deviation payoff or a low value of the discount factor), the players need to play a non-cooperative action in that period, which *reduces* the on-path payoff, i.e., the incentive to follow the equilibrium strategy. Therefore the players need to be more patient under perturbation than in the deterministic case.[5]

By contrast, in our outside option model, Player 1 can choose between playing the game forever and stopping. Thus, when the perturbation creates a difficulty to cooperate (a high outside option), it does not mean that Player 1 must endure the low

[3] As α increases more, e.g., $\alpha > 1.6$, then $7 + \delta W > 7 + \delta \frac{4}{1-\delta}$ so that after defection, Player 1 wants to wait for the better option. However, $5 + \delta V > 7 + \delta W$ holds so that the stochastic cooperation continues to be an equilibrium behavior.

[4] McAdams (2007) considers volatility in the stage game payoffs with a fixed outside payoff. The volatility in his model is not a perturbation but a state-dependent in-game payoffs to see how future in the game affects the current incentives.

[5] A similar argument is noted in Mailath and Samuelson (2006), p.176-177.

payoff of a non-cooperative action. The difficulty to cooperate means that stopping the game is more beneficial, and hence she can take that option to *increase* the on-path payoff, i.e., the incentive to follow the equilibrium strategy. Therefore lower discount factors are sufficient to sustain the equilibrium than the ones under the unique option.

In summary, we have shown that there are perturbations that can increase the value of repeated cooperation, and this occurs naturally in the context of outside options in endogenously repeated games. Our model fits to many economic situations such as employment relationships and repeat purchase situations. Therefore perturbations of outside options should be carefully treated in those applications.

The outline of the paper is as follows. In Section 2, we formulate the basic one-sided outside option model. In Section 3, we first analyze the benchmark model of single-deterministic option in Section 3.1. The main analysis of the paper is Sections 3.2 and 3.3, where stochastic option model is analyzed and the effect of the mean and spread of options are derived. In Section 4, we consider two important extensions of continuum of outside options and two-sided outside options. Section 5 concludes the paper.

2 One-sided Outside-option Model

Consider a two-player dynamic game as follows. Time is discrete and denoted as $t = 1, 2, \dots$ but the game continues endogenously. At the beginning of period $t = 1, 2, \dots$ as long as the game continues, two players, called Player 1 and Player 2, simultaneously choose one of the actions from the set $\{C, D\}$ of the Prisoner's Dilemma. The action C is interpreted as a cooperative action and the action D is interpreted as a defective action. We denote the symmetric payoffs associated with each action profile as[6]: $u(C,C) = c$, $u(C,D) = \ell$, $u(D,C) = g$, $u(D,D) = d$ with the ordering $g > c > d > \ell$ and $2c > g + \ell$. See Table 2. The latter inequality implies that (C,C) is efficient among correlated action profiles.

After observing this period's action profile, an outside option becomes available to Player 1. The game continues to the next period if and only if Player 1 does not take an outside option. Each player maximizes the total expected discounted payoff[7] with

[6] The first coordinate is the player's own action, and the second coordinate is the opponent's action.

[7] Alternatively one can assume that the players maximize the average payoffs without changing the qualitative results.

P1 \ P2	C	D
C	c, c	ℓ, g
D	g, ℓ	d, d

Table 2: General Prisoner's Dilemma

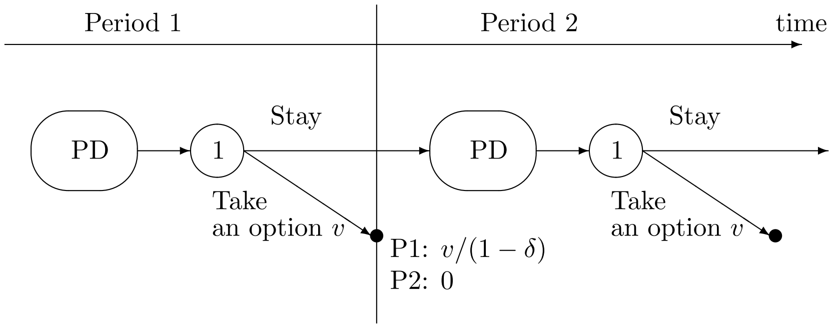


Figure 1: Outline of the Single Option Game

a common discount factor $\delta \in (0, 1)$.

The value of an option is of the form $x/(1-\delta)$ and there can be multiple values of x among which one option becomes available at the end of each period. An interpretation of this formulation is that the same payoff $\{x, x, \ldots\}$ obtains forever after the game ends. We identify an outside option by the average payoff x.

Player 2 receives payoff only from the Prisoner's Dilemma as long as the game continues and Player 2 does not have the ability to end the game, as in the ordinary repeated games. Let us also assume that $d \geqq 0$ which implies that Player 2's "outside payoff" 0 is not better than the payoff from (D, D). This simplifies our analysis by making Player 2's deviation not relevant. (To be precise, the qualitative result does not change as long as Player 2's outside payoff is not greater than Player 1's average outside option.) Figure 1 shows the outline of the dynamic game when there is a single outside option $v/(1-\delta)$ for Player 1 in any period.

If Player 1 takes an outside option $v/(1-\delta)$ at the end of T-th period, her total payoff is

$$\sum_{t=1}^{T} \delta^{t-1} u(a(t)) + \delta^T \frac{v}{1-\delta},$$

while Player 2's total payoff is

$$\sum_{t=1}^{T} \delta^{t-1} u(a(t)),$$

where $a(t)$ is the action profile in t-th period of the repeated Prisoner's Dilemma.

We assume that all actions are observable to the players. Therefore, in period $t \geqq 2$, players can base their actions on the history of past action profiles. The game is of complete information. As the equilibrium concept, we use subgame perfect equilibrium (SPE henceforth).

There are many economic situations that fit into this model. For example, we can interpret the model as a buyer-seller model such that Player 1 is a buyer, Player 2 is a seller, C is an honest action in transactions and D is a dishonest action. We can also interpret the model as an employment relationship such that Player 1 is a worker and Player 2 is a firm.

3 Deterministic vs. Stochastic Outside Options

We investigate the range of δ in which repeated mutual cooperation of (C, C) is sustained as long as possible, under a variety of structures of the outside options. If the maximal equilibrium punishment does not sustain the on-path action profile, no other punishment would, by the same logic as the optimal penal code in Abreu (1988). Therefore, without loss of generality we consider the following type of strategy combinations, which we call "simple trigger strategy" combinations. Note that Player 1's optimal exit strategy varies depending on the structure of outside options.

Cooperation phase: If the history is empty or does not have D, play (C, C) and Player 1 uses an optimal exit strategy given that (C, C) is repeated as long as the game continues.

Punishment phase: If the history contains D, play (D, D) and Player 1 uses an optimal exit strategy given that (D, D) is repeated as long as the game continues.

Let us justify our focus on (C, C) on the play path in three ways. First, under some circumstances, we cannot lower the minimum discount factor δ by including other action profiles in the play path. For example, we may want to include (D, C) occasionally to give Player 1 more incentive to follow the strategy. However, if the payoff of Player 2 is too low under (D, C), then Player 2's incentive to play the strategy increases the necessary δ. Such parameter range is characterized in Appendix A of Fujiwara-Greve and Yasuda (2009). The motivating example in the Introduction is in fact such a case. Moreover when we consider two-sided outside options (see Section 4.1), clearly an asymmetric action profile would make it more difficult to sustain.

Second, our interest is how the structure of outside options affects the lower bound, not the absolute value of the lower bound. The qualitative results such as the mean effect and the perturbation effect (Section 3.3) do not depend on the focus on (C, C). For example, by including (D, C), the value of the play path for Player 1 increases so that her minimum discount factor decreases, but still there is a sufficient size of spread between the outside options that reduces the discount factor further. Therefore the conclusion that perturbation reduces the minimum discount factor is valid.

Third, the pure action profile (C, C) has a clear meaning in economics, for example, honest transactions, contribution to public good and absence of moral hazard. Then, it is an important class of strategies to be analyzed.

3.1. Single Deterministic Option Case

As the benchmark, we first consider the case of a unique outside option v for Player 1. That is, at the end of each period, the same option becomes available, which gives $v/(1-\delta)$ in total after the exit (Figure 1). We assume that $d < v < c$, since this is the only interesting case.[8] Under this assumption, the optimal exit strategy of Player 1 in the cooperation phase is never to exit and the one in the punishment phase is to take the option at the first opportunity. Therefore, the play path of the simple trigger strategy combination is the eternal cooperation. Let us find the range of δ that makes the simple trigger strategy with the eternal cooperation a SPE.

Recall that in the ordinary repeated Prisoner's Dilemma with discounting, the eternal cooperation is sustained by the simple trigger strategy without the exit option if and only if

$$\begin{aligned} \frac{c}{1-\delta} &\geqq g + \frac{\delta d}{1-\delta} \\ \iff \delta &\geqq \frac{g-c}{g-d} =: \underline{\delta}. \end{aligned}$$

In the presence of the outside option v, Player 1 does not deviate in the cooperation

[8]If $v \geqq c$, then Player 1 would never cooperate since she will exit immediately. If $v \leqq d$, then the game is effectively a repeated game since even in the punishment phase Player 1 would not exit.

phase if and only if

$$\frac{c}{1-\delta} \geqq g + \frac{\delta v}{1-\delta} \tag{1}$$

$$\iff \delta \geqq \frac{g-c}{g-v} =: \delta_1^D(v), \tag{2}$$

and Player 2 does not deviate in the cooperation phase if and only if

$$\frac{c}{1-\delta} \geqq g \iff \delta \geqq \frac{g-c}{g} =: \delta_2^D.$$

Let $\delta^D(v) = \max\{\delta_1^D(v), \delta_2^D\}$. Then the simple trigger strategy combination is a SPE if and only if $\delta \geqq \delta^D(v)$. Moreover, $v > d$ implies that $\delta_1^D(v) > \underline{\delta}$, and $d \geqq 0$ implies that $\delta_2^D \leqq \underline{\delta}$. Hence $\delta^D(v) = \delta_1^D(v) > \underline{\delta}$. This means that, for any $\delta \in [\underline{\delta}, \delta^D(v))$, the existence of an outside option, greater than the mutual defection payoff d, makes the eternal cooperation impossible, while it was possible if the game were an ordinary repeated Prisoner's Dilemma. It is also easy to see that $\delta^D(v)$ is increasing in v, implying that better outside option makes it harder to cooperate. Since $\lim_{v \to c} \delta^D(v) = 1$, the range of δ that sustains the eternal cooperation shrinks to the empty set, as the outside option approaches to c.

Proposition 1. *For any $v \in (d, c)$, the eternal cooperation is sustained as the outcome of a SPE if and only if $\delta \geqq \delta^D(v) > \underline{\delta}$. Hence, for any $\delta \in [\underline{\delta}, \delta^D(v))$, the eternal cooperation cannot be sustained in the outside option model, while it is sustainable in the ordinary repeated Prisoner's Dilemma.*

In addition, Fujiwara-Greve and Yasuda (2009) shows that if the value of the outside option fluctuates over time but is deterministic, then the eternal cooperation falls apart by backward induction if there is a (known) period in which the outside option value exceeds the inverse of $\delta^D(v)$.

3.2. Binary Stochastic Options

Let us turn to stochastic outside options. The randomness can be interpreted several ways, such as subjective uncertainty, external perturbation, or a draw from a distribution of options. To make a comparison with the single option model, we fix the mean of the outside options as $v \in (d, c)$ throughout this section.

The stochasticity of the options changes both the value of cooperation phase and the value of the punishment phase so that in addition to the eternal cooperation and

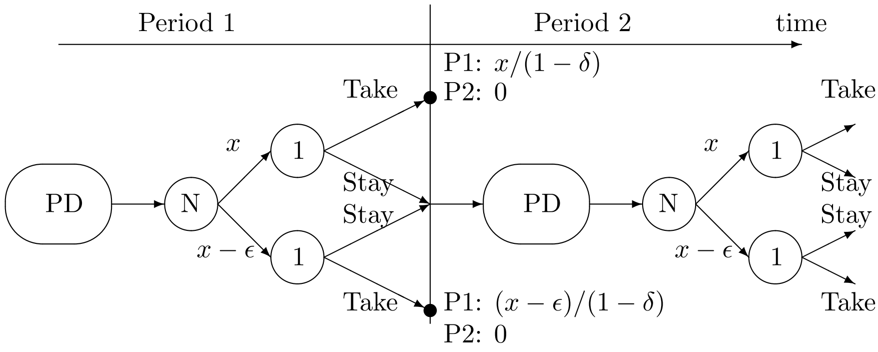


Figure 2: Outline of the Binary Stochastic Option Game

no cooperation, the stochastic cooperation (cooperation until a stochastic end of the game) may become the play path. Then it is possible that the volatility of the payoffs changes the on-path play from no cooperation to the stochastic cooperation, as we discussed in Introduction.

Even though we fix the mean of the outside options, there are many ways to formulate a stochastic distribution of binary options. We focus on a class of binary options such that $x/(1-\delta)$ and $(x-\epsilon)/(1-\delta)$ are the two possible values that realize with probability p and $1-p$ respectively. This formulation allows us to vary the mean and the spread ϵ of the options independently to see their effects on cooperation.[9] The example in the Introduction is a special case where $p = 0.5$, $\epsilon = 2\alpha$ and $v = x - \frac{\epsilon}{2}$. The outline of the new dynamic game is depicted in Figure 2.

Note that $v = px + (1-p)(x-\epsilon)$ implies that

$$x(v, \epsilon) = v + \epsilon(1-p).$$

Thus, when the spread ϵ changes, the option values must change accordingly to keep the mean as v. To simplify the notation, we often suppress the parameters and write x when there is no danger of confusion.

We first derive optimal exit strategies for Player 1, when repeated (C, C) or (D, D) is expected as long as the game continues. First, suppose that (C, C) is expected as long as the game continues. If Player 1 takes any of the options to quit immediately,

[9] At the end of Section 3.3, we briefly discuss an alternative formulation of the binary stochastic options and robustness of the results. Fujiwara-Greve and Yasuda (2009) analyzes the alternative formulation.

the expected continuation payoff is $p\frac{x}{1-\delta} + (1-p)\frac{x-\epsilon}{1-\delta} = \frac{v}{1-\delta}$, which is less than $\frac{c}{1-\delta}$ by the assumption. Hence taking any outside option is not an optimal exit strategy when (C, C) is expected as long as the game continues.

It remains to clarify when not taking any option is better than taking only the good option x. Let V be the continuation payoff measured right before an option realizes, when Player 1 plans to take only x and stays in the game when the lower option is realized. It satisfies the following recursive equation.

$$V = p\frac{x}{1-\delta} + (1-p)\big[c + \delta V\big].$$

To explain, the first term is the total discounted payoff when the better option realizes, which occurs with probability p. The second term is the total discounted payoff when the worse option realizes, because in that case Player 1 stays in the game, receives c in the next period and then faces the same situation as now.

V can be explicitly solved as

$$V(v, \epsilon) = \frac{1}{1-(1-p)\delta}\Big[p\frac{x}{1-\delta} + (1-p)c\Big]. \tag{3}$$

We made clear that V depends on (v, ϵ), since x depends on (v, ϵ).

Remark 1. *For any $\delta \in (0, 1)$ and any $p \in (0, 1)$,*

$$\begin{aligned}\frac{c}{1-\delta} \geqq V(v, \epsilon) \quad &\Longleftrightarrow \quad \{1-(1-p)\delta\}c \geqq (1-\delta)\Big[p\frac{x}{1-\delta} + (1-p)c\Big] \\ &\Longleftrightarrow \quad \{1-(1-p)\delta - (1-\delta)(1-p)\}c \geqq px \\ &\Longleftrightarrow \quad c \geqq x. \end{aligned} \tag{4}$$

Therefore, in ex ante values, rejecting any option (and getting $c/(1-\delta)$) in the cooperation phase is optimal if and only if $c \geqq x$.

Note that the optimization problem *after* a realization of an option gives exactly the same condition. After a realization, Player 1 compares taking the realized option with the (ex ante) continuation payoff in the future, which is either $c/(1-\delta)$ or $c+\delta V(v, \epsilon)$. Thus, Player 1 facing a realized option of x compares $\frac{x}{1-\delta}$ with $\max\{\frac{c}{1-\delta}, c+\delta V(v, \epsilon)\}$. If $x \leqq c$, then $\frac{x}{1-\delta} \leqq \frac{c}{1-\delta} = \max\{\frac{c}{1-\delta}, c + \delta V(v, \epsilon)\}$ by Remark 1. Thus she would not take x. If $x > c$, then the following Remark 2 shows that $\frac{x}{1-\delta} > c + \delta V(v, \epsilon) = \max\{\frac{c}{1-\delta}, c + \delta V(v, \epsilon)\}$. Thus she should take x now.

Remark 2. *For any $\delta \in (0,1)$ and any $p \in (0,1)$,*

$$\begin{aligned} & c + \delta V(v,\epsilon) - \frac{x}{1-\delta} \\ = \; & c + \delta p \frac{x}{1-\delta} + \delta(1-p)\{c + \delta V(v,\epsilon)\} - \{x + \delta p \frac{x}{1-\delta} + \delta(1-p)\frac{x}{1-\delta}\} \\ = \; & (c-x) + \delta(1-p)\{c + \delta V(v,\epsilon) - \frac{x}{1-\delta}\}. \end{aligned}$$

Therefore, $c + \delta V(v,\epsilon) \geqq \frac{x}{1-\delta} \iff c \geqq x$.

By the formulation, $x - \epsilon = v - \epsilon p < c$. Hence after a realization of $x - \epsilon$, Player 1 would not take the low option.

In summary we have the following characterization of the optimal exit strategy in the cooperation phase.

Lemma 1. *For any $\epsilon > 0$, when (C, C) is expected as long as the game continues, not taking any outside option is the optimal exit strategy for Player 1 if $c \geqq x$, and taking only the good option x is optimal otherwise.*

Analogously, suppose that repeated (D, D) is expected as long as the game continues. Clearly, the optimal exit strategy is either to take both options or to wait for the better option, since rejecting both options gives her only $d/(1-\delta)$, which is strictly worse than taking both options. Let W be the continuation payoff measured right before an option realizes and when Player 1 plans to take only the better option x. By an analogous argument as for V, W satisfies

$$W = p\frac{x}{1-\delta} + (1-p)\big[d + \delta W\big].$$

Thus,

$$W(v,\epsilon) = \frac{1}{1-(1-p)\delta}\Big[p\frac{x}{1-\delta} + (1-p)d\Big]. \tag{5}$$

Waiting for the good option x is better than taking any option to quit immediately if and only if

$$\begin{aligned} W(v,\epsilon) \geqq \frac{v}{1-\delta} \quad & \iff \quad (1-\delta)\Big[p\frac{x}{1-\delta} + (1-p)d\Big] \geqq \{1-(1-p)\delta\}v \\ & \iff \quad \delta(1-p)(v-d) \geqq v - d - p\{x-d\} \\ & \iff \quad \delta \geqq \frac{v-d-\epsilon p}{v-d}, \end{aligned} \tag{6}$$

where the last equivalence uses $x = v + \epsilon(1-p)$. Let $\delta^P(v,\epsilon) = \frac{v-d-\epsilon p}{v-d}$. The superscript P stands for the punishment phase. In sum, we have the following characterization of the optimal exit strategy in the punishment phase.

Lemma 2. *For any $\epsilon > 0$, when (D, D) is expected as long as the game continues, taking only the good outside option of x is the optimal exit strategy for Player 1 if $\delta \geqq \max\{\delta^P(v,\epsilon), 0\}$, and taking any outside option is optimal otherwise.*

We now find the lower bound of the discount factor δ to sustain repeated mutual cooperation *as long as possible*, using the simple trigger strategy combination. Lemma 1 implies that if the good option x does not exceed the mutual cooperation payoff, i.e., $c \geqq x$, then the eternal cooperation is possible, while if $x > c$, then only the stochastic cooperation is possible. We show that in the former case, the lower bound (denoted as $\delta^E(v,\epsilon)$) of discount factors that sustain the eternal cooperation is never lower than $\delta^D(v)$, while in the latter case the lower bound (denoted as $\delta^S(v,\epsilon)$) can be less than $\delta^D(v)$. Below we give graphical explanations, and formal proofs of propositions are in the Appendix.

Proposition 2. *Take any $\epsilon \in (0, \frac{c-v}{1-p}]$ so that $c \geqq x$. Let $\delta^{cW}(v,\epsilon)$ be the solution to*

$$\frac{c}{1-\delta} = g + \delta W(v,\epsilon).$$

The eternal cooperation is sustained if and only if δ is not less than

$$\delta^E(v,\epsilon) := \max\{\delta^D(v), \delta^{cW}(v,\epsilon)\}.$$

Since $\delta^E(v,\epsilon) \geqq \delta^D(v)$, the eternal cooperation is not easier under perturbation than under the single option v.

Proof: See Appendix.

The intuition is as follows. Since $c \geqq x$, the optimal value of the cooperation phase is $c/(1-\delta)$ by Lemma 1. The value of the optimal one-shot deviation is

$$\max\{g + \delta\frac{v}{1-\delta},\ g + \delta W(v,\epsilon)\}.$$

As δ increases from 0 to 1, both of these values increase, but the value of the cooperation phase is more convex than the optimal one-shot deviation value. (See Figure 3, which is drawn under the parameter values $(g, c, d, \ell, v, p, \epsilon) = (10, 8, 3, 2, 5, 0.5, 3.5)$.)

Recall that $\delta^D(v)$ equalizes $\frac{c}{1-\delta}$ and $g + \delta\frac{v}{1-\delta}$. We have two cases; wether $\delta^D(v) \leqq \delta^P(v,\epsilon)$ or not. If $\delta^D(v) \leqq \delta^P(v,\epsilon)$, then the optimal cooperation value $c/(1-\delta)$ intersects with the optimal deviation value when the latter is $g + \delta v/(1-\delta)$, so that

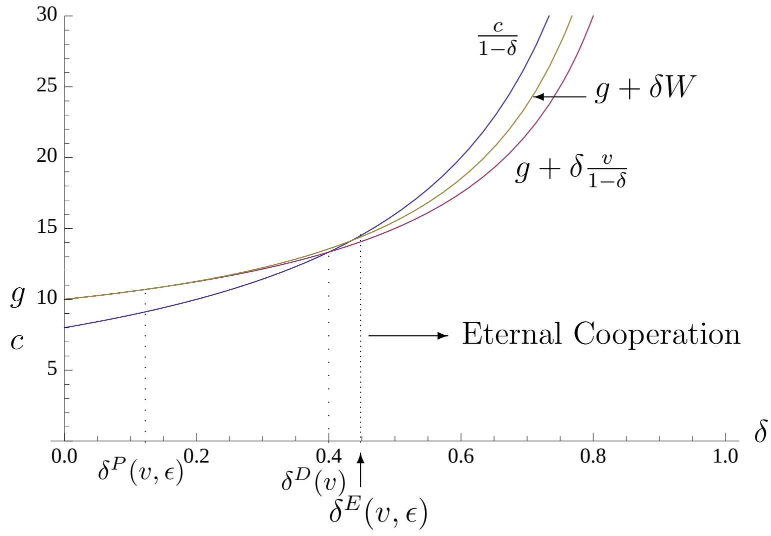


Figure 3: Less Cooperation under Stochastic Outside Options ($c \geqq x$)

Player 1 follows the eternal cooperation if and only if $\delta \geqq \delta^D(v)$. On the other hand, if $\delta^P(v, \epsilon) < \delta^D(v)$, then $c/(1-\delta)$ must exceed $g+\delta W(v, \epsilon)$ in order to induce Player 1 to follow the eternal cooperation, as shown in Figure 3. The lower bound to the discount factors in this case is $\delta^{cW}(v, \epsilon)$.

In summary, when $c \geqq x$, the eternal cooperation is sustained[10] if and only if

$$\delta \geqq \max\{\delta^D(v), \delta^{cW}(v, \epsilon)\} =: \delta^E(v, \epsilon)$$

and this lower bound is never smaller than $\delta^D(v)$.

Two remarks are in order. First, it is possible to sustain the stochastic cooperation (to cooperate until x realizes) under $c \geqq x$ as well, but higher δ is needed because the value of the stochastic cooperation is smaller than $c/(1-\delta)$. Second, $\delta^D(v) \leqq \delta^P(v, \epsilon)$ if and only if $\epsilon p \leqq (v-d)(c-v)/(g-v)$. This means that as ϵ becomes larger, $\delta^P(v, \epsilon) \geqq \delta^D(v)$ is violated so that the lower bound $\delta^E(v, \epsilon)$ becomes $\delta^{cW}(v, \epsilon)$ which is strictly greater than $\delta^D(v)$. That is, greater spread (more volatility) makes the eternal cooperation more difficult. This perturbation effect is extensively studied in Section 3.3.

Next, we consider even greater ϵ so that $x > c$. In this case only the stochastic cooperation can be sustained.

[10]To be precise, Player 2's deviations must be checked. This is done in the formal proof.

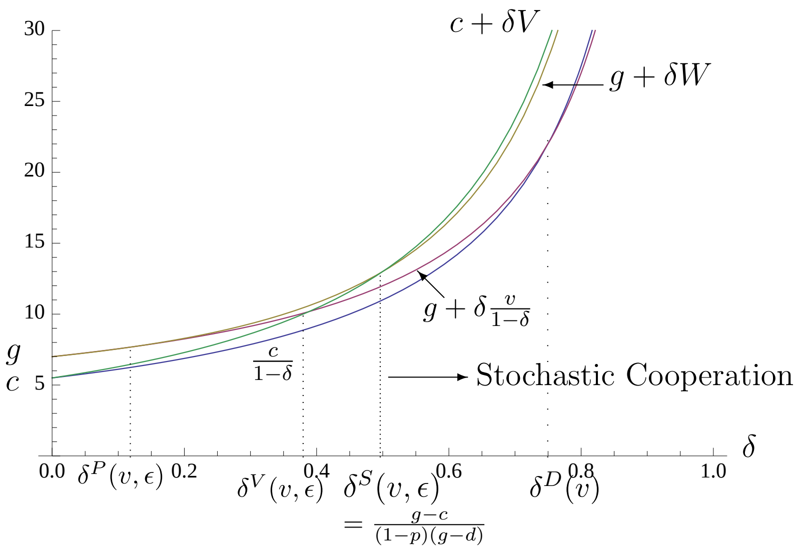


Figure 4: More Cooperation under Stochastic Outside Options ($x > c$)

Proposition 3. *Take any* $\epsilon > \frac{c-v}{1-p}$ *so that* $x > c$. *Let* $\delta^V(v,\epsilon)$ *be the solution to*

$$c + \delta V(v,\epsilon) = g + \delta\frac{v}{1-\delta}.$$

Then the stochastic cooperation is sustained if and only if δ *is not less than*

$$\delta^S(v,\epsilon) := \max\{\delta^V(v,\epsilon), \frac{g-c}{(1-p)(g-d)}\}.$$

Moreover, $\delta^S(v,\epsilon) < \delta^D(v)$ *if and only if* $v > pg + (1-p)d$. *That is, when* g *is sufficiently small, cooperation becomes easier under perturbation than under the single option* v.

Proof: See Appendix.

The intuition is as follows. Notice that when $x > c$, (4) implies that, for any $\delta > 0$,

$$c + \delta V(v,\epsilon) > \frac{c}{1-\delta}. \tag{7}$$

That is, as Figure 4 shows,[11] the value of stochastic cooperation is uniformly greater than the value of eternal cooperation. There are two possibilities of how $c + \delta V(v,\epsilon)$ intersects with the optimal deviation value, $\max\{g+\delta W(v,\epsilon), g+\delta\frac{v}{1-\delta}\}$. If $c+\delta V(v,\epsilon)$

[11]The parameter values are $(g, c, d, \ell, v, p, \epsilon) = (7, 5.5, 1, 0.1, 5, 0.5, 7)$.

intersects with the optimal deviation value when the latter is $g + \delta\frac{v}{1-\delta}$, the intersection is $\delta^V(v,\epsilon)$. Note also that (7) implies that $\delta^V(v,\epsilon) < \delta^D(v)$.

If $c + \delta V(v,\epsilon)$ intersects with the optimal deviation value when the latter is $g + \delta W(v,\epsilon)$, the intersection is computed as follows. From (3) and (5),

$$c + \delta V(v,\epsilon) = g + \delta W(v,\epsilon) \iff \delta = \frac{g-c}{(1-p)(g-d)}. \tag{8}$$

Therefore, the stochastic cooperation is sustained if and only if

$$\delta \geqq \max\{\delta^V(v,\epsilon), \frac{g-c}{(1-p)(g-d)}\} =: \delta^S(v,\epsilon).$$

Since $\delta^V(v,\epsilon) < \delta^D(v)$, the lower bound $\delta^S(v,\epsilon)$ is strictly smaller than $\delta^D(v)$ if and only if $\frac{g-c}{(1-p)(g-d)} < \delta^D(v)$ which is equivalent to $v > pg + (1-p)d$.

In words, when the one-shot deviation gain g is not too large, even though δ is not sufficient for cooperation under the unique option v, stochastic cooperation can be sustained under perturbation. This is because the increase in the value of stochastic cooperation, $c + \delta V(v,\epsilon)$ as compared to $c/(1-\delta)$, is greater than that of the deviation value, $g + \delta W(v,\epsilon)$ as compared to $g + \delta v/(1-\delta)$.

Note, however, that $\delta^S(v,\epsilon) \geqq \frac{g-c}{(1-p)(g-d)} > \frac{g-c}{g-d} = \underline{\delta}$ for any $p > 0$. Therefore, mutual cooperation is still more difficult than in the ordinary repeated game. The "locked-in" feature of repeated games is a strong device to enforce cooperation.

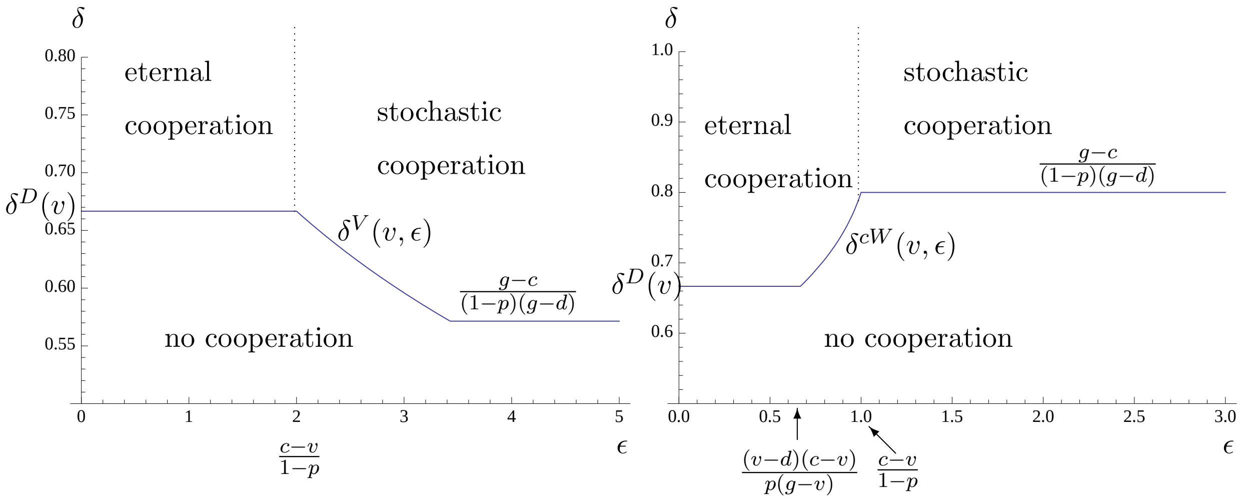
3.3. Mean Effect and Perturbation Effect

In this subsection we analyze the effect of changes in the mean v and the spread ϵ. The increase of the mean v increases the option value of the punishment phase more than that of the cooperation phase. Therefore the increase of the mean makes cooperation more difficult, just like in the unique option case.

Corollary 1. *Given ϵ, $\delta^E(v,\epsilon)$ is increasing in v, and $\delta^S(v,\epsilon)$ is non-decreasing in v. That is, as the mean of the outside options increases, cooperation becomes more difficult.*

Proof : See Appendix.

By contrast, the perturbation effect of ϵ is more complex, since it only affects the value when Player 1 wants to wait for the good option, i.e., given δ and v, the increase



5(a): $v > pg + (1-p)d$ 5(b): $v \leqq pg + (1-p)d$

Figure 5: Perturbation Effect

of ϵ increases $V(v,\epsilon)$ and $W(v,\epsilon)$ only. As Figure 5 shows, depending on whether $v > pg + (1-p)d$ or not, the perturbation effect of ϵ is different. It is easy to see that $v > pg + (1-p)d$ implies that $\delta^D(v) < \delta^P(v,\epsilon)$. In this case, greater spread (larger perturbation) is good for cooperation, which is depicted in Figure 5(a).

Corollary 2. *Assume that $v > pg + (1-p)d$. For any $\epsilon > 0$, the lower bound to the discount factors that sustain eternal/stochastic cooperation is non-increasing in ϵ. That is, as the spread of the option values increases, cooperation becomes weakly easier.*

Proof : See Appendix.

The intuition is as follows. Let us start with small $\epsilon \leqq \frac{c-v}{1-p}$. In this region either eternal cooperation or no cooperation is sustained. Since $v > pg+(1-p)d$ implies that $\delta^D(v) < \delta^P(v,\epsilon)$, the relevant lower bound is $\delta^E(v,\epsilon) = \delta^D(v)$, which is constant over ϵ. As ϵ increases so that $\epsilon > \frac{c-v}{1-p}$ holds, i.e., $x > c$, either stochastic cooperation or no cooperation is sustained. For medium value of ϵ, the optimal deviation value is $g+\delta\frac{v}{1-\delta}$ and thus $\delta^V(v,\epsilon)$ is the lower bound. It is easy to see that $\delta^V(v,\epsilon)$ is decreasing in ϵ since the on-path value $c + \delta V(v,\epsilon)$ is increasing while the deviation value $g + \delta\frac{v}{1-\delta}$ is constant. For large value of ϵ, the optimal deviation value is $g + \delta W(v,\epsilon)$ so that the constant $\frac{g-c}{(1-p)(g-d)}$ is the lower bound.

As we look at Figure 5(a) horizontally, we see that given a mid-range of δ, no cooperation becomes stochastic cooperation as ϵ increases. Thus for this case cooperation is enhanced by the perturbation.

By contrast, when $v \leqq pg + (1-p)d$, then $\delta^E(v,\epsilon) = \delta^D(v)$ for small $\epsilon < \frac{(v-d)(c-v)}{p(g-v)}$ and $\delta^E(v,\epsilon) = \delta^{cW}(v,\epsilon)$ for mid-range $\epsilon \in [\frac{(v-d)(c-v)}{p(g-v)}, \frac{c-v}{1-p})$. As ϵ increases further, the lower bound becomes $\frac{g-c}{(1-p)(g-d)}$. See Figure 5(b). Thus in this case the lower bound is non-decreasing in ϵ, i.e., perturbation is bad for cooperation.

The ordinary repeated game literature looks only at the vertical axis of Figure 5, where $\epsilon = 0$, and the case of $v = d$. By adding the dimension of ϵ, we enlarged the scope of the analysis and found the positive effect of payoff perturbation.

Our result is different from the effect of stochastic discount factor (Dal Bó, 2007), which affects both the cooperation phase value and the punishment phase value, and that of stochastic payoffs in Rotemberg and Saloner (1986). As we discussed in Introduction, their results can be interpreted as the eternal cooperation being more difficult under volatility. We have provided a third source of volatility via the outside options and expanded the notion of "repeated cooperation" to include not only the eternal cooperation but also the stochastic cooperation. Then we can show that in some cases cooperation is enhanced under more volatility.

Yasuda and Fujiwara-Greve (2009) shows a similar result for ordinary repeated games with perturbed payoffs. Essentially, if the volatility of the payoffs takes the form that stopping cooperation in that period is beneficial, then players can still *selectively cooperate* in some periods, even if they cannot cooperate under no perturbation. If the stage game allows this, the lower bound to the discount factors is less than the one without perturbation.

Finally we comment on the effect of p. Since p changes both the probability of the good option $x = v + \epsilon(1-p)$ as well as its value, the effect of p is clearly not monotonic. The most interesting case is when p is very small so that the stochastic cooperation is almost the eternal cooperation. The relevant lower bound is

$$\delta^S(v,\epsilon) := \max\{\delta^V(v,\epsilon), \frac{g-c}{(1-p)(g-d)}\}.$$

As p converges to 0, $\frac{g-c}{(1-p)(g-d)}$ converges to $\underline{\delta}$, but $\delta^V(v,\epsilon)$ converges to $\delta^D(v)$. Therefore, when the probability of exit by taking the good option becomes negligible, the model converges to the unique option case.[12]

[12] This is in fact dependent on our formulation of the stochastic options. In Fujiwara-Greve and Yasuda (2009), we give an alternative formulation of binary stochastic options such that the good option is defind as $x = \frac{v-v^-}{p} + v^-$ where $v^- < v$ is a fixed value of the low option. In this case, as p converges to 0, the model converges to the ordinary repeated game because in both the cooperation

4 Extensions

4.1. Continuum of Outside Options

The binary distribution model illustrates well the essence of the effect of stochastic outside options on the cooperation within the repeated game. However, it is of some theoretical interest how the model and results extend to a case with a continuum of outside options, which is more standard in some economic models such as search models. We show that the stochastic cooperation is sustained under lower discount factors than those of the single deterministic option, even under a continuum of outside options.

Assume that Player 1 has a continuum of outside options with the support $[\underline{v}, \overline{v}]$. That is, at the end of each period, an option $x \in [\underline{v}, \overline{v}]$ realizes for Player 1 and if she takes this option, she receives the total payoff of $\frac{x}{1-\delta}$ after exit. Let F be the (differentiable) cumulative distribution function of the outside options and f be its density function. Assume, as before, that the mean outside option $v := \int_{\underline{v}}^{\overline{v}} xf(x)dx$ is strictly between d and c.

If Player 1 takes an option of value x, then she would also take any option greater than x. Hence the optimal exit strategy is a *reservation strategy*: Player 1 takes any outside option not less than a certain level r, where r is called the reservation level. Suppose that as long as Player 1 is in the game, she can receive u from the Prisoner's Dilemma, where u can be either c or d. Let $U(u, r)$ be the value, at the end of a period before a stochastic outside option realizes, and when Player 1 takes any option not less than $r \in [\underline{v}, \overline{v}]$. It satisfies the following recursive equation:

$$U(u, r) = \int_{r}^{\overline{v}} \frac{x}{1-\delta} f(x)dx + F(r)\{u + \delta U(u, r)\}. \tag{9}$$

By differentiation of (9) with respect to r, we have

$$\begin{aligned} \frac{\partial U(u, r)}{\partial r} &= -\frac{r}{1-\delta} f(r) + f(r)\{u + \delta U(u, r)\} + \delta F(r) \frac{\partial U(u, r)}{\partial r}, \\ \iff \quad \frac{\partial U(u, r)}{\partial r} &= \frac{f(r)}{1 - \delta F(r)} \big[u - \frac{r}{1-\delta} + \delta U(u, r)\big]. \end{aligned}$$

phase and the punishment phase Player 1 waits for the good option but it hardly arrives. Therefore the lower bound converges to $\underline{\delta}$. Note, however, that the effect of v and ϵ are robust under this formulation.

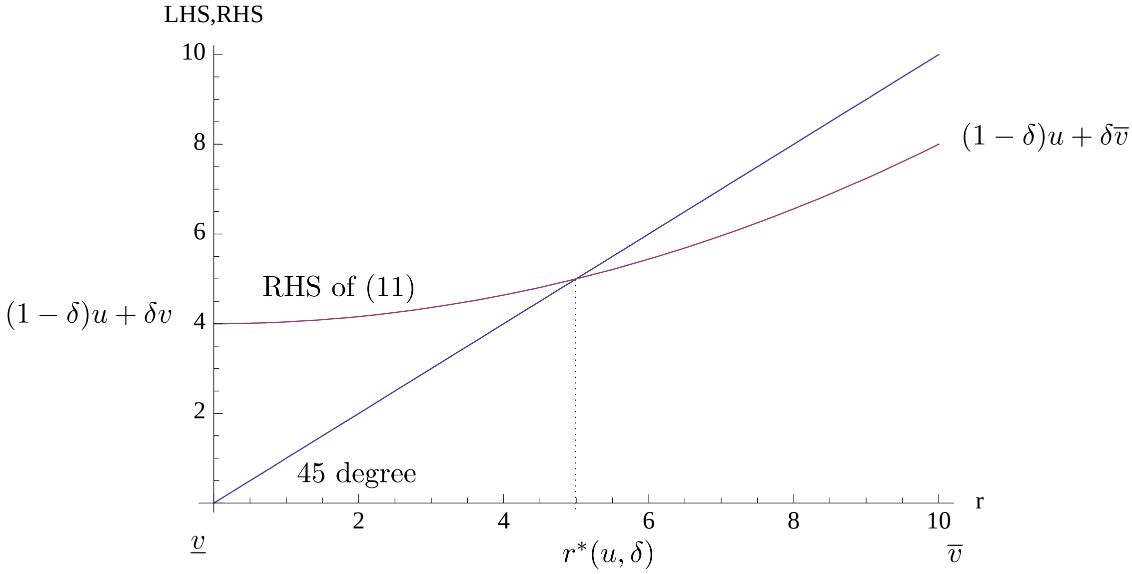


Figure 6: Optimal Reservation Level ($F=$UNIF$[0,10]$, $\delta=0.8$, $u=0$)

The optimal reservation level, denoted as $r^*(u,\delta)$, is the solution to $\frac{\partial U(u,r)}{\partial r}=0$ (since the second order condition holds), that is,

$$\frac{r^*(u,\delta)}{1-\delta}=u+\delta U(u,r^*(u,\delta)). \tag{10}$$

This means that the optimal reservation level of the outside options is exactly where Player 1 is indifferent between taking it and not taking it. (9) and (10) imply that

$$\begin{aligned} & \frac{r^*(u,\delta)}{1-\delta}=u+\delta U(u,r^*(u,\delta)) \\ \iff \quad & r^*(u,\delta)=(1-\delta)u+\delta(1-\delta)\Big\{\int_{r^*(u,\delta)}^{\overline{v}}\frac{x}{1-\delta}f(x)dx+F(r^*(u,\delta))\frac{r^*(u,\delta)}{1-\delta}\Big\}. \end{aligned}$$

Hence, for any $\delta\in(0,1)$ and any $u=c,d$, the optimal reservation level $r^*(u,\delta)$ is the solution to the following equation:

$$r=(1-\delta)u+\delta\int_r^{\overline{v}}xf(x)dx+\delta F(r)r. \tag{11}$$

By differentiation it is straightforward to show that the RHS of (11) is a monotone increasing function of r, taking value from $(1-\delta)u+\delta\underline{v}$ to $(1-\delta)u+\delta\overline{v}$. Figure 6 illustrates this property. Therefore, in the cooperation phase where $u=c$, the optimal reservation level $r^*(c,\delta)$ is less than $\overline{v}$ if and only if $(1-\delta)c+\delta\overline{v}>\overline{v}$, which is equivalent to $\overline{v}>c$.

Lemma 3. *When (C, C) is expected as long as the game continues, the optimal exit strategy for Player 1 is to not to take any outside option if $c \geqq \overline{v}$, and to take any outside option not less than $r^*(c, \delta)$ otherwise.*

In the following we focus on the stochastic cooperation, i.e., we assume that $\overline{v} > c$ and give a sufficient condition under which the lower bound of the discount factors that sustain the stochastic cooperation is less than $\delta^D(v)$.

The equation (11) implies that in the punishment phase when $u = d$, the optimal reservation level is $\underline{v}$ (that is, it is optimal to exit by taking any option) if and only if

$$\underline{v} \geqq (1-\delta)d + \delta \int_{\underline{v}}^{\overline{v}} x f(x) dx + \delta F(\underline{v})\underline{v} = (1-\delta)d + \delta v,$$

which is equivalent to

$$\delta \leqq \frac{\underline{v} - d}{v - d}.$$

This corresponds to $\delta \leqq \delta^P(v, \epsilon)$ for the binary case. Note that

$$\delta^D(v) \leqq \frac{\underline{v} - d}{v - d} \iff (\underline{v} - d)(g - v) \geqq (g - c)(v - d). \tag{12}$$

Therefore, if (12) holds, then the on-path value $c + \delta U(c, r^*(c, \delta))$ (which is strictly greater than $\frac{c}{1-\delta}$ under the assumption $\overline{v} > c$) intersects with the one-shot deviation value when this is $g + \delta \frac{v}{1-\delta}$. Hence the lower bound of the discount factors that deter Player 1's deviation is strictly less than $\delta^D(v)$. In addition, if we impose an extra condition, Player 2 does not deviate either.

Proposition 4. *Assume that $\overline{v} > c$, (12), and $v > \{1 - F(c)\}g$. Let δ^F be the lower bound of δ that sustains the stochastic cooperation under the continuum of outside options with the distribution F. Then $\delta^F < \delta^D(v)$.*

Proof: See Appendix.

We have shown that there is a case of continuum outside options in which the stochastic cooperation is sustained under lower discount factors than those under the unique option.

4.2 Two-sided outside options

Let us extend the model so that Player 2 also has non-negligible outside options. Then there are two new aspects to be clarified. First, the rule of termination must

P1 \ P2	
Stay	Continue
Exit	End

3(a): One-sided Option for Player 1

P1 \ P2	Stay	Exit
Stay	Continue	End
Exit	End	End

3(b): Two-sided Option with Unilateral Ending Rule

Table 3: Game Continuation Patterns

be carefully specified. Second, under stochastic options, the optimal exit decision may become a coordination game, because if the other player is waiting for the good option, one may also wait for one and vice versa, even under independent distributions. Otherwise the analysis is similar to the one-sided option model.

When both players can choose to take outside options, the rule of termination of a repeated game becomes relevant. The unilateral ending rule assumed in the one-sided option model (Table 3(a)) has a specific meaning in the two-sided option model that the repeated game ends if and only if *at least* one player chooses to exit (Table 3(b)). There is an intermediate case of two-sided option model in which both players must agree to end the game, but in that case it is straightforward to prove that any equilibrium outcome of ordinary repeated game can be sustained.[13] Therefore the essentially different models from ordinary repeated games are the one-sided option model and two-sided option model with the unilateral ending rule. Moreover, the unilateral ending rule is the most commonly analyzed rule (e.g., Gosh and Ray, 1996; Kranton, 1996a,b; Fujiwara-Greve, 2002 and Fujiwara-Greve and Okuno-Fujiwara, 2009) and describes well situations such as joint ventures and lender-borrower relationships.

The unique option case is rather simple. Let $v_1, v_2 \in (d, c)$ be the outside options for Player 1 and Player 2 respectively. By the same argument as in Section 3.1, Player i would not play C if $\delta < \delta_i^D(v_i) =: \frac{g-c}{g-v_i}$. The range of discount factors that sustains mutual cooperation is $\delta \geqq \max\{\delta_1^D(v_1), \delta_2^D(v_2)\}$. Hence the lower bound is not less than the one when only one of the players has a unique option v_i.

[13]For example, repeated (C, C) can be achieved by the following strategy combination if two players must agree to end the game: Play C and do not take outside options as long as no one played D. If someone played D in the past, play D and do not take outside options. Since one player cannot unilaterally end the game to escape, the strategy combination is a subgame perfect equilibrium if and only if the usual grim-trigger strategy combination is a subgame perfect equilibrium in the ordinary repeated Prisoner's Dilemma.

P1 \ P2	Stay	Take only x
Stay	$\frac{c}{1-\delta}$, $\frac{c}{1-\delta}$	$c+\delta V''$, $c+\delta V(v,\epsilon)$
Take only x	$c+\delta V(v,\epsilon)$, $c+\delta V''$	$c+\delta V'$, $c+\delta V'$

Table 4: Payoff Combinations in the Cooperation Phase

Let us now turn to the stochastic options, where Player 1 and Player 2 independently draw outside options from the same i.i.d. distribution such that $x/(1-\delta)$ obtains with probability p and $(x-\epsilon)/(1-\delta)$ obtains with probability $(1-p)$. The symmetry is assumed to make the comparison easy with the one-sided option case. Under the independent draws, a player may take an outside option to terminate the game, even when the other player does not want to take own option, so that the game ends with a different probability and the payoff becomes different from the one in the one-sided outside option case.

First, consider the cooperation phase. There are two candidates for an optimal exit strategy given that (C, C) continues as long as someone terminates the game: not to take any option ("stay") or take only the better option x ("take only x"). Depending on the other player's exit strategy, the value of these exit strategies are different. If the other player is not taking any option, you are as if Player 1 in the one-sided option model. Hence not taking any option gives you $c/(1-\delta)$, while taking only x gives you $c+\delta V(v,\epsilon)$. If both players take x but reject $x-\epsilon$, the ex ante continuation value just before the option realizes (denoted as V') satisfies the following recursive equation.

$$V' = p\frac{x}{1-\delta} + p(1-p)\frac{x-\epsilon}{1-\delta} + (1-p)^2(c+\delta V'). \tag{13}$$

This is because with probability $p(1-p)$, one's option turns out to be $x-\epsilon$ but the partner's turned out to be x, in which case the game ends and one ends up with the low option. Similarly, let V'' be the ex ante continuation value just before the option realizes, when you do not take any option but your partner takes x. It satisfies

$$V'' = p\frac{v}{1-\delta} + (1-p)(c+\delta V''). \tag{14}$$

To explain, with probability p, your partner terminates the game by taking x but in that case you receive on average $v/(1-\delta)$. With probability $1-p$ the game continues.

The total expected payoffs of the two players, of various combinations of exit strategies, are summarized in Table 4. Lemma 4 (b) shows when $x > c$, both $c+\delta V(v,\epsilon) >$

P1 \ P2	Exit	Take only x
Exit	$\frac{v}{1-\delta}$, $\frac{v}{1-\delta}$	$\frac{v}{1-\delta}$, $\frac{v}{1-\delta}$
Take only x	$\frac{v}{1-\delta}$, $\frac{v}{1-\delta}$	W', W'

Table 5: Continuation Value Combinations in the Punishment Phase

$c/(1-\delta)$ and $c+\delta V' > c+\delta V''$ hold so that there is a unique equilibrium of (Take only x, Take only x).

By contrast, if $c \geqq x$, then there can be two equilibria of (Stay, Stay) and (Take only x, Take only x) or a unique equilibrium of (Stay, Stay), depending on whether $V' \geqq V''$ or not. This is a new feature of the two-sided option model that the exit strategies may constitute a coordination game.

Lemma 4. *(a) For any* (δ, p, v, ϵ), $V(v,\epsilon) > V'$ *and* $V(v,\epsilon) > V''$.

(b) For any (δ, p, v, ϵ), $V' \geqq V''$ *if and only if* $x - c + \delta p(1-p)\frac{\epsilon}{1-\delta} \geqq 0$.

Proof: See Appendix.

Although it is possible to have two equilibria when $c \geqq x$, Lemma 4 (a) and $c/(1-\delta) \geqq c+\delta V(v,\epsilon)$ together imply that (Stay, Stay) payoff dominates (Take only x, Take only x). Therefore, we focus on the eternal cooperation for $c \geqq x$ and the stochastic cooperation for $c < x$, as in the one-sided option model.

Next, we clarify equilibria in the punishment phase. When (D, D) is expected until someone terminates the game, there are two possible optimal exit strategies: to take only the good option x or to take any option to exit immediately. Let W' be the ex ante continuation value just before an option realizes, when both players take only x. It satisfies essentially the same recursive structure as V':

$$W' = p\frac{x}{1-\delta} + p(1-p)\frac{x-\epsilon}{1-\delta} + (1-p)^2(d+\delta W'). \tag{15}$$

If your partner exits immediately by taking any option, by the unilateral ending rule, your choice of exit strategy is irrelevant and the continuation value is $v/(1-\delta)$. Thus, in the subgames after a deviation, the continuation value comparison is simpler than the cooperation phase, as Table 5 shows.

We can show that whether $v/(1-\delta)$ is greater than W' or not is again determined by the same critical value of $\delta^P(v,\epsilon)$, which equalizes $W(v,\epsilon)$ with $v/(1-\delta)$. (See (6).) Moreover, the optimal deviation value is never greater than that of the one-sided option model.

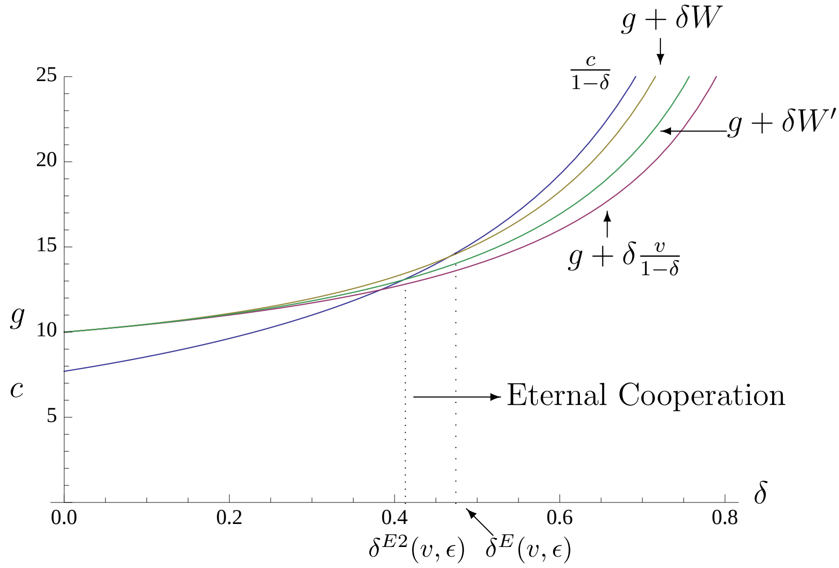


Figure 7: More Cooperation under Two-Sided Outside Options ($c \geqq x$)

Lemma 5. *For any (v, ϵ), the one-shot deviation values are ordered as follows.*

$$\delta \leqq \delta^P(v,\epsilon) \quad \Rightarrow \quad g + \delta\frac{v}{1-\delta} \geqq g + \delta W' \geqq g + \delta W(v,\epsilon);$$
$$\delta^P(v,\epsilon) \leqq \delta \quad \Rightarrow \quad g + \delta W(v,\epsilon) \geqq g + \delta W' \geqq g + \delta\frac{v}{1-\delta}.$$

Proof: See Appendix.

This means that, when $\delta \leqq \delta^P(v,\epsilon)$, the optimal deviation value is the same across the one-sided and two-sided option models, but, when $\delta \geqq \delta^P(v,\epsilon)$, the optimal deviation value is less under the two-sided options.

Therefore, when $c \geqq x$ so that the eternal cooperation is to be sustained, the decrease of the punishment phase value makes the eternal cooperation easier in the two-side option model, i.e., the lower bound $\delta^{E2}(v,\epsilon)$ that sustains eternal cooperation is never greater than $\delta^E(v,\epsilon)$ for the one-sided option model. See Figure 7.[14]

By contrast, when $x > c$, so that the stochastic cooperation is to be sustained, both the on-path value $c + \delta V'$ and the punishment phase value, $\max\{g + \delta W', g + \delta\frac{v}{1-\delta}\}$, are reduced under the two-sided option model. Let $\delta^{V'}(v,\epsilon)$ be the solution to

$$c + \delta V' = g + \delta\frac{v}{1-\delta}.$$

[14]The parameter values are $(g, c, d, \ell, v, p, \epsilon) = (10, 7.7, 0.5, 0.1, 4, 0.5, 7)$.

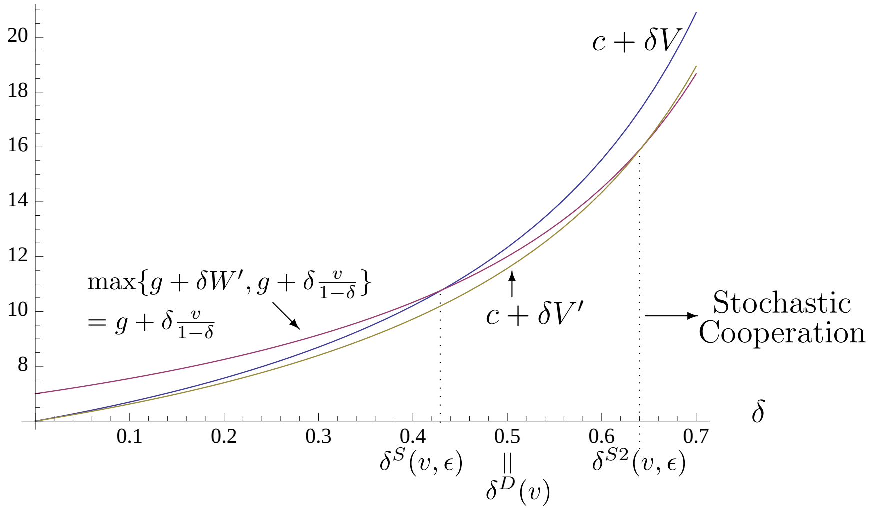


Figure 8: Less Cooperation under Two-Sided Outside Options ($x > c$)

Then $c + \delta V > c + \delta V'$ implies that $\delta^V(v,\epsilon) < \delta^{V'}(v,\epsilon)$. On the other hand,

$$c + \delta V' \geqq g + \delta W' \iff \delta \geqq \frac{g-c}{(1-p)^2(g-d)} > \frac{g-c}{(1-p)(g-d)}.$$

Therefore, the lower bound $\delta^{S2}(v,\epsilon) := \max\{\delta^{V'}(v,\epsilon), \frac{g-c}{(1-p)^2(g-d)}\}$ that sustains stochastic cooperation under the two-sided options is greater than $\delta^S(v,\epsilon)$. (See Figure 8.[15])

We can summarize the results for two-sided options as follows.

Proposition 5. *Case 1: For* $\epsilon \leqq \frac{c-v}{1-p}$ *so that* $c \geqq x$. *Let* $\delta^{E2}(v,\epsilon)$ *be the lower bound of the discount factors that sustain the eternal cooperation under two-sided outside options. Then* $\delta^{E2}(v,\epsilon) \leqq \delta^E(v,\epsilon)$.
Case 2: For $\epsilon > \frac{c-v}{1-p}$ *so that* $x > c$. *Let* $\delta^{S2}(v,\epsilon)$ *be the lower bound of the discount factors that sustain the stochastic cooperation under two-sided outside options. Then* $\delta^{S2}(v,\epsilon) > \delta^S(v,\epsilon)$.

For the two-sided option model, we only need to check one player's optimization, which is analogous to the one in Propositions 2 and 3 and is explained above. Therefore the proof is omitted.

Even if the realizations of outside options for two players are not completely independent, as long as there is a positive probability that a player end up with the low option, the same result follows. Essentially, the effect of perturbation is weakened relative to the one-sided case, because a player may not be able to wait for a good option

[15] The parameter combination is $(g, c, d, \ell, v, p, \epsilon) = (7, 6, 0.2, 0.1, 5, 1.5)$.

when she wanted to, which reduces the value of options. The weaker effect of perturbation is in fact two-fold: the eternal cooperation becomes easier and the stochastic cooperation becomes more difficult than the one-sided option case.

5 Concluding Remarks

Our result can be summarized in three points. First, payoff perturbation (stochastic outside options) may enhance cooperation, which is a new insight. In the literature of ordinary repeated games, only infinitely-repeated cooperation has been analyzed and thus payoff perturbation has negative effect, since perturbation increases the temptation to deviate at some point. However if we extend the notion of "repeated cooperation" to include stochastic repetition of cooperation and perturbation of outside options is considered, a player may want to wait for a good option, which makes her more patient.

Second, the effect of the mean and the spread of outside options are quite different. The effect of the mean is monotone and negative in the sense that higher discount factors are needed to sustain cooperation as the mean of the outside options increases. By contrast, the effect of the spread between options is more complex, as shown in Figure 5. For mid-range discount factors and when the deviation gain is not too large, the increase of the spread between options enhances cooperation. Therefore, the option *structure* is important, and "fixing the outside option as a single value" is not without loss of generality.

Third, two-sided outside options weaken the effect of stochastic outside options. The reason is as follows. If both players can end the game unilaterally, the game ends more frequently and the option value is reduced, since the partner may end the game when one is faced with a low option. This makes the cooperation easier if the punishment phase payoff is reduced but more difficult if the cooperation phase payoff is reduced.

Finally, let us mention future directions. Although the main concern in the present paper is to analyze the sustainability of mutual cooperation under perturbations, it should also be of interest to characterize the set of equilibrium payoffs. Especially, comparative static of the equilibrium payoff sets with respect to the mean value and/or the spread of the outside options has great importance. As we showed in Section 3.3, increased volatility of ϵ can make Player 1's cooperation easier, which implies that the set of equilibrium payoffs need not be monotonically decreasing (in the sense of set inclusion) as the outside options increase. This non-monotonicity of equilibrium payoffs may have significant implications to applications, for example in policy effects.[16]

In addition, we would like to point out that there is a wide scope of important applications of our model, for example, employment relationships, buyer-seller relationships and marriages. An important implication from our result is that specifications of what players may receive outside of the game, such as potential wage offers and reservation utilities, can have significant effects on their in-game strategic incentives. This finding stands in sharp contrast to the traditional modeling approach in dynamic games and contracting where the outside structure of a game is often assumed to be fixed. We believe that our model can provide meaningful insights and implications for many applications.

APPENDIX: Proofs

For notational simplicity, we abbreviate $V(v,\epsilon)$ and $W(v,\epsilon)$ as V and W except in the proof of Corollary 1 and 2.

Proof of Proposition 2: Recall that

$$\frac{c}{1-\delta} \geqq g+\delta\frac{v}{1-\delta} \iff \delta \geqq \delta^D(v), \tag{2}$$

$$g+\delta W \geqq g+\delta\frac{v}{1-\delta} \iff \delta \geqq \delta^P(v,\epsilon). \tag{6}$$

We also show that the on-path value function $c/(1-\delta)$ exceeds the deviation value $g+\delta W$ for any δ above some critical δ. By computation,

$$\begin{aligned} & \frac{c}{1-\delta} \geqq g+\delta W = g+\delta\frac{p\frac{x}{1-\delta}+(1-p)d}{1-(1-p)\delta}, \\ \iff \quad & h(\delta) := -\delta^2(1-p)(g-d)+\delta\{(1-p)(g-c)+g-px-(1-p)d\}-(g-c) \geqq 0. \end{aligned}$$

Notice that h is quadratic in δ, $h(0)=-(g-c)<0$ and $h(1)=p(c-x)\geqq 0$ by the assumption. Therefore there exists $\delta^{cW}(v,\epsilon)\in(0,1]$ such that

$$\frac{c}{1-\delta} \geqq g+\delta W \iff \delta \geqq \delta^{cW}(v,\epsilon). \tag{16}$$

[16]There is a different non-monotonicity result. In a class of games called exhaustible resource games, Dutta (1995) showed that the first-best outcome is sustainable under a mid-range discount factor but not under high discount factors.

Next we divide into two cases.

Case 1: $0 < \epsilon \leqq \frac{(v-d)(c-v)}{(1-p)(g-v)}$, i.e., $\delta^D(v) \leqq \delta^P(v,\epsilon)$.

In this case, for any $\delta \leqq \delta^D(v)$, $g + \delta\frac{v}{1-\delta} \geqq g + \delta W$. Thus, as δ increases from 0 to 1, the on-path value $\frac{c}{1-\delta}$ intersects with $g + \delta W$ before it does with $g + \delta\frac{v}{1-\delta}$, i.e., $\delta^{cW}(v,\epsilon) \leqq \delta^D(v)$. Hence

$$\frac{c}{1-\delta} \geqq \max\{g + \delta W, g + \delta\frac{v}{1-\delta}\} \iff \delta \geqq \delta^D(v) = \max\{\delta^D(v), \delta^{cW}(v,\epsilon)\}.$$

Case 2: $\frac{(v-d)(c-v)}{(1-p)(g-v)} < \epsilon$, i.e., $\delta^P(v,\epsilon) < \delta^D(v)$.

In this case, when the on-path value $c/(1-\delta)$ intersects with $g + \delta\frac{v}{1-\delta}$ (at $\delta^D(v)$), the optimal one-shot deviation value is in fact $g+\delta W$. Thus the on-path value function intersects with the optimal one-shot deviation value $\max\{g + \delta W, g + \delta\frac{v}{1-\delta}\}$ when the latter is $g + \delta W$ (see Figure 3), at $\delta^{cW}(v,\epsilon)$. Therefore

$$\frac{c}{1-\delta} \geqq \max\{g + \delta W, g + \delta\frac{v}{1-\delta}\} \iff \delta \geqq \delta^{cW}(v,\epsilon).$$

Note also that since $g+\delta W > g+\delta\frac{v}{1-\delta}$ at $\delta = \delta^D(v)$ and since $\frac{c}{1-\delta}$ is strictly increasing in δ, $\delta^D(v) < \delta^{cW}(v,\epsilon)$ in this case.

Finally, Player 2's deviation value changes depending on whether Player 1 exits immediately or not after seeing a deviation. If Player 1 exits immediately, i.e., if $\max\{W, \frac{v}{1-\delta}\} = \frac{v}{1-\delta}$, Player 2's deviation value is $g + \delta \cdot 0$.

If Player 1 waits for the good option in the punishment phase, i.e., if $\max\{W, \frac{v}{1-\delta}\} = W$, then Player 2's deviation value is increased to

$$g + (1-p)\delta d + (1-p)^2\delta^2 d + \cdots = g + \frac{(1-p)\delta d}{1-(1-p)\delta}.$$

In this case Player 2 does not deviate in the cooperation phase if and only if

$$\frac{c}{1-\delta} \geqq g + \delta\frac{(1-p)d}{1-(1-p)\delta}. \tag{17}$$

Notice that the RHS of (17) is strictly smaller than $g + \delta W = g + \delta\frac{p\frac{x}{1-\delta}+(1-p)d}{1-(1-p)\delta}$, since $x > 0$.

Therefore, Player 1's optimal deviation value $\max\{g + \delta W, g + \delta\frac{v}{1-\delta}\}$ is always greater than Player 2' s deviation value and thus as long as Player 1 does not deviate, Player 2 does not. □

Proof of Proposition 3: First, we show that there exists a unique $\delta^V(v,\epsilon) \in (0, \delta^D(v))$ such that

$$\delta \gtreqqless \delta^V(v,\epsilon) \iff c + \delta V \gtreqqless g + \delta \frac{v}{1-\delta}.$$

(See Figure 4.) Let

$$\begin{aligned} h'(\delta) &:= (1-\delta)\{1-(1-p)\delta\}\{c+\delta V - g - \delta\frac{v}{1-\delta}\} \\ &= -(g-v)(1-p)\delta^2 + \delta\{(1-p)g + g - c + px - v)\} - (g-c). \end{aligned}$$

Then

$$c + \delta V \gtreqqless g + \delta \frac{v}{1-\delta} \iff h'(\delta) \gtreqqless 0.$$

Since $h'(\delta)$ is a concave, quadratic function of δ, $h(0) = -(g-c) < 0$, and $h(1) = p(x-v) > 0$, there exists a unique $\delta^V(v,\epsilon) \in (0,1)$ such that $h'(\delta) \gtreqqless 0$ if and only if $\delta \gtreqqless \delta^V(v,\epsilon)$. To show that $\delta^V(v,\epsilon) < \delta^D(v)$, plug in $\delta = \delta^D(v)$ into h' and we get

$$h'(\delta^D(v)) = \frac{(g-c)p(x-c)}{g-v} > 0,$$

by the assumption that $x > c$. Thus $\delta^D(v) > \delta^V(v,\epsilon)$.

Second, we show that Player 1 does not deviate for any $\delta \geqq \max\{\delta^V(v,\epsilon), \frac{g-c}{(1-p)(g-d)}\}$. Recall that from (8), we have that

$$c + \delta V \gtreqqless g + \delta W \iff \delta \gtreqqless \frac{g-c}{(1-p)(g-d)}.$$

Similar to the proof of Proposition 2, when $c+\delta V$ intersects with the optimal deviation value $\max\{g+\delta W, g+\delta\frac{v}{1-\delta}\}$ at $\delta \leqq \delta^P(v,\epsilon)$, the latter is $g+\delta\frac{v}{1-\delta}$ and $\delta^V(v,\epsilon) \geqq \frac{g-c}{(1-p)(g-d)}$. Then $\max\{g+\delta W, g+\delta\frac{v}{1-\delta}\} = g+\delta\frac{v}{1-\delta}$ implies that Player 1 does not deviate if and only if $\delta \geqq \delta^V(v,\epsilon)$. If $c+\delta V$ intersects with the optimal deviation value at $\delta > \delta^P(v,\epsilon)$, the latter is $g+\delta W$ and $\delta^V(v,\epsilon) < \frac{g-c}{(1-p)(g-d)}$. Thus Player 1 does not deviate if and only if $\delta \geqq \frac{g-c}{(1-p)(g-d)}$ in this case. (See Figure 4.)

Next, consider Player 2. Let V_2 be Player 2's continuation payoff before an option for Player 1 realizes during the cooperation phase. Since Player 1 exits with probability p, it satisfies

$$V_2 = (1-p)\{c + \delta V_2\} + p \cdot 0.$$

Thus $V_2 = \frac{(1-p)c}{1-(1-p)\delta}$ and the on-path value for Player 2 is $c + \delta V_2 = \frac{c}{1-(1-p)\delta}$.

If he deviates, Player 1 exits immediately if $v/(1-\delta) \geqq W$ or equivalently $\delta \leqq \delta^P(v,\epsilon)$, and Player 1 waits for the good option otherwise. Let W_2 be the continuation payoff during the punishment phase for Player 2, when Player 1 waits for the good option. It satisfies

$$W_2 = (1-p)\{d + \delta W_2\} + p \cdot 0,$$

so that $W_2 = (1-p)d/\{1-(1-p)\delta\}$. Hence the one-shot deviation value for Player 2 is

$$\begin{cases} g + \delta \cdot 0 & \text{if } \delta \leqq \delta^P(v,\epsilon) \\ g + \delta W_2 & \text{if } \delta^P(v,\epsilon) \leqq \delta. \end{cases}$$

Since $d \geqq 0$, it suffices to show that the lower bound of δ that satisfies

$$c + \delta V_2 \geqq g + \delta W_2$$

is less than $\delta^D(v)$. Note that the payoff structure is similar for Player 2 and Player 1;

$$V_2 - W_2 = (1-p)\{c + \delta V_2\} + p \cdot 0 - (1-p)\{d + \delta W_2\} - p \cdot 0,$$

and

$$V - W = (1-p)\{c + \delta V\} + p \cdot \frac{x}{1-\delta} - (1-p)\{d + \delta W\} - p \cdot \frac{x}{1-\delta}.$$

Hence $V_2 - W_2 = V - W$ and since

$$c + \delta V \geqq g + \delta W \iff \delta \geqq \frac{g-c}{(1-p)(g-d)},$$

Player 2 does not deviate if and only if $\delta \geqq \frac{g-c}{(1-p)(g-d)}$.

Therefore $\delta^S(v,\epsilon) = \max\{\delta^V(v,\epsilon), \frac{g-c}{(1-p)(g-d)}\}$ is the lower bound of the discount factor that sustains the stochastic cooperation. Finally, note that $\frac{g-c}{(1-p)(g-d)} < \delta^D(v)$ if and only if $v > pg + (1-p)d$. Thus $\delta^S(v,\epsilon) < \delta^D(v)$ if and only if $v > pg + (1-p)d$. □

Proof of Corollary 1: When v is small so that $x \leqq c$ (i.e., the on-path value is $c/(1-\delta)$), only the deviation value $\max\{g + \delta W(v,\epsilon), g + \delta\frac{v}{1-\delta}\}$ increases as v increases. Hence $\delta^E(v,\epsilon)$ is increasing in v.

As v becomes larger so that $x > c$, the on-path value $c + \delta V(v,\epsilon)$ also increases as v increases, so that we need to see the relative change between the on-path value and the punishment phase. Recall that the critical value $\frac{g-c}{(1-p)(g-d)}$ is independent of v. Recall the definition of $\delta^V(v,\epsilon)$;

$$c + \delta V(v,\epsilon) = g + \delta\frac{v}{1-\delta} \iff \frac{\delta(1-p)\{(1-\delta)(c-v) + \epsilon p\}}{(1-\delta)\{1-(1-p)\delta\}} = g - c.$$

Notice that the LHS of the second equality is increasing in δ and decreasing in v. Therefore, $\delta^V(v,\epsilon)$ is increasing in v, and thus $\max\{\delta^V(v,\epsilon), \frac{g-c}{(1-p)(g-d)}\}$ is non-decreasing in v. □

Proof of Corollary 2: Recall that $c \geqq x$ if and only if $\epsilon \leqq \frac{c-v}{1-p}$. Thus there are two regions of ϵ, $0 < \epsilon \leqq \frac{c-v}{1-p}$ and $\frac{c-v}{1-p} < \epsilon$ to be distinguished. In the former, the relevant lower bound to the discount factors is $\delta^E(v,\epsilon) = \max\{\delta^D(v), \delta^{cW}(v,\epsilon)\}$. Notice that

$$v > pg + (1-p)d \iff \frac{c-v}{1-p} < \frac{(v-d)(c-v)}{p(g-v)}.$$

Hence $\epsilon \leqq \frac{c-v}{1-p}$ implies that $\epsilon < \frac{(v-d)(c-v)}{p(g-v)}$, which is equivalent to $\delta^D(v) < \delta^P(v,\epsilon)$. Therefore when $v > pg + (1-p)d$, $\delta^E(v,\epsilon) = \delta^D(v)$, which is constant over ϵ.

In the latter interval $\frac{c-v}{1-p} < \epsilon$, the relevant lower bound to the discount factors is $\delta^S(v,\epsilon) = \max\{\delta^V(v,\epsilon), \frac{g-c}{(1-p)(g-d)}\}$.

We show that $\delta^V(v,\epsilon)$ is decreasing in ϵ. Recall the definition of $\delta^V(v,\epsilon)$,

$$c + \delta V(v,\epsilon) = g + \delta\frac{v}{1-\delta}.$$

As ϵ increases, the LHS cooperation value increases while the RHS deviation value is the same. Since the LHS intersects with the RHS from below (see Figure 4), the intersection δ^V decreases, as ϵ increases. □

Proof of Proposition 4: It suffices to prove that Player 2 does not deviate under $\delta \geqq \delta^F$. Recall that Player 1 exits with probability $1 - F(r^*(d,\delta))$ if the optimal reservation level is $r^*(d,\delta)$. Hence Player 2's deviation value is

$$\begin{cases} g + \delta \cdot 0 & \text{if } \delta \leqq \frac{\underline{v}-d}{v-d} \\ g + \delta\frac{d}{1-\delta F(r^*(d,\delta))} & \text{if } \frac{\underline{v}-d}{v-d} \leqq \delta. \end{cases}$$

Player 2's total expected payoff in the cooperation phase is

$$\frac{c}{1-\delta F(r^*(c,\delta))}.$$

Since we have assumed that $\delta^D(v) \leqq \frac{\underline{v}-d}{v-d}$, it suffices to show that the smallest δ that satisfies

$$\frac{c}{1-\delta F(r^*(c,\delta))} \geqq g \tag{18}$$

is not more than $\delta^D(v)$. By rearrangement, (18) is equivalent to

$$\delta F(r^*(c,\delta))g \geqq g - c.$$

We first prove that $c < r^*(c,\delta)$. Notice that $\overline{v} > c$ is equivalent to

$$\begin{aligned} & \int_c^{\overline{v}} (x-c)f(x)dx > 0 \\ \iff & \int_c^{\overline{v}} xf(x)dx + F(c)c > c \\ \iff & (1-\delta)c + \delta \int_c^{\overline{v}} xf(x)dx + \delta F(c)c > c. \end{aligned}$$

This implies that at $r = c$, the RHS of (11) is above the 45 degree line. Hence the intersection with the 45 degree line (which is $r^*(c,\delta)$) is greater than c for any δ. (See Figure 6.) Therefore we also have that $F(c) < F(r^*(c,\delta))$ for any δ, and thus $v > \{1 - F(c)\}g$ implies that

$$\delta F(r^*(c,\delta))g > \delta F(c)g > \delta(g - v).$$

Second, note that when $\delta = \delta^D(v)$, $\delta(g - v) = g - c$. Therefore at $\delta = \delta^D(v)$,

$$\delta F(r^*(c,\delta))g > g - c,$$

and $\delta F(r^*(c,\delta))g$ is uniformly greater than $\delta F(c)g$ for any $\delta \in (0,1)$. Thus there exists $\delta^{F2} < \delta^D(v)$ such that for any $\delta \geqq \delta^{F2}$, Player 2 does not deviate. Let δ^{F1} be the bound for Player 1, then as shown in the text $\delta^{F1} < \delta^D(v)$ as well. Let $\delta^F = \max\{\delta^{F1}, \delta^{F2}\}$ then this is the lower bound that sustains the stochastic cooperation and is strictly smaller than $\delta^D(v)$. □

Proof of Lemma 4: (a) Recall that

$$\begin{aligned} V &= p\frac{x}{1-\delta} + (1-p)(c + \delta V) \\ V' &= p\frac{x}{1-\delta} + (1-p)\{p\frac{x-\epsilon}{1-\delta} + (1-p)(c + \delta V')\} \end{aligned}$$

By subtracting both sides, we have

$$V - V' = (1-p)[\delta(V - V') + p\{(c + \delta V') - \frac{x-\epsilon}{1-\delta}\}].$$

Since $c + \delta V' > \frac{x-\epsilon}{1-\delta}$, we have that $V > V'$.

Analogously, subtract V'' from V and we get

$$V - V'' = p(\frac{x}{1-\delta} - \frac{v}{1-\delta}) + \delta(1-p)(V - V'').$$

Since $x > v$, we have that $V > V''$.

(b) Rearrange V' so that

$$V' = p\frac{v}{1-\delta} + (1-p)\{p\frac{x}{1-\delta} + (1-p)(c + \delta V')\}.$$

By subtracting V'' from this, we have

$$V' - V'' = (1-p)p\{\frac{x}{1-\delta} - (c + \delta V')\} + \delta(1-p)(V' - V'').$$

Therefore $V' \geqq V''$ if and only if $\frac{x}{1-\delta} \geqq c + \delta V'$. Let us compare:

$$\begin{aligned} \frac{x}{1-\delta} &= x + \delta\{p\frac{x}{1-\delta} + p(1-p)\frac{x}{1-\delta} + (1-p)^2\frac{x}{1-\delta}\} \\ c + \delta V' &= c + \delta\{p\frac{x}{1-\delta} + p(1-p)\frac{x-\epsilon}{1-\delta} + (1-p)^2(c + \delta V')\}. \end{aligned}$$

By subtracting both sides, we have that

$$\frac{x}{1-\delta} - (c + \delta V') = x - c + \delta[p(1-p)\frac{\epsilon}{1-\delta} + (1-p)^2\{\frac{x}{1-\delta} - (c + \delta V')\}].$$

Hence $V' \geqq V'' \iff \frac{x}{1-\delta} \geqq c + \delta V'$ if and only if $x - c + \delta p(1-p)\frac{\epsilon}{1-\delta} \geqq 0$. □

Proof of Lemma 5: From (15), we have

$$W' = \frac{p\frac{x}{1-\delta} + p(1-p)\frac{x-\epsilon}{1-\delta} + (1-p)^2 d}{1 - (1-p)^2\delta}.$$

Using $x = v + \epsilon(1-p)$, we have

$$(\frac{v}{1-\delta} - W')(1-\delta)\{1 - (1-p)^2\delta\} = (1-p)^2\{v - d - \epsilon p - (v-d)\delta\},$$

so that

$$\frac{v}{1-\delta} \geqq W' \iff \delta \leqq \delta^P(v, \epsilon). \tag{19}$$

Moreover, by comparing (5) and (15);

$$\begin{aligned} W &= p\frac{x}{1-\delta} + p(1-p)(d + \delta W) + (1-p)^2(d + \delta W) \\ W' &= p\frac{x}{1-\delta} + p(1-p)\frac{x-\epsilon}{1-\delta} + (1-p)^2(d + \delta W') \\ \Rightarrow W - W' &= p(1-p)(d + \delta W - \frac{x-\epsilon}{1-\delta}). \end{aligned}$$

Therefore $W \geqq W'$ if and only if $d + \delta W \geqq \frac{x-\epsilon}{1-\delta}$. By rearrangements,

$$
\begin{aligned}
& d + \delta W \geqq \frac{x-\epsilon}{1-\delta} \\
\iff \quad & (1-p)(d+\delta W) \geqq (1-p)\frac{x-\epsilon}{1-\delta} \\
\iff \quad & p\frac{x}{1-\delta} + (1-p)(d+\delta W) \geqq p\frac{x}{1-\delta} + (1-p)\frac{x-\epsilon}{1-\delta} \\
\iff \quad & W \geqq \frac{v}{1-\delta} \\
\iff \quad & \delta \geqq \delta^P(v,\epsilon).
\end{aligned}
$$

Combined with (19), we have that

$$
\begin{aligned}
\delta \leqq \delta^P(v,\epsilon) \quad & \Rightarrow \quad g + \delta\frac{v}{1-\delta} \geqq g + \delta W' \geqq g + \delta W; \\
\delta^P(v,\epsilon) \leqq \delta \quad & \Rightarrow \quad g + \delta W \geqq g + \delta W' \geqq g + \delta\frac{v}{1-\delta}.
\end{aligned}
$$

□

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